

Avoidance Control on Time Scales*

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Abstract

We consider dynamic systems on time scales under the control of two agents. One of the agents desires to keep the state of the system out of a given set regardless of the other agent's actions. Leitmann's avoidance conditions are proved to be valid for dynamic systems evolving on an arbitrary time scale.

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1 Introduction

The concept of avoidance control was introduced and investigated by Leitmann and his co-authors in a series of papers [1–7]. For a recent survey on avoidance control we refer the reader to [8]. The goal of this paper is to initiate the study of avoidance control on an arbitrary time scale.

Analysis on time scales is nowadays recognized as the right tool to unify and extend the seemingly disparate fields of discrete-time dynamical systems

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and continuous-time dynamical systems [9–11]. In the present paper when the time scale is fixed to be the real line, then one gets the classical results of Leitmann and Skowroński [6]; when the time scale is chosen to be the integers, one gets analogous results in the discrete-time case. Moreover, other models of time can be considered, and an illustrative example of the results of the paper is considered for the periodic case.

We consider delta-dynamical systems on time scales under the control of two agents: the evader and the pursuer. The evader desires a strategy guaranteeing that no trajectory of the delta-system, emanating from outside the antitarget, intersects that set, no matter what strategy the pursuer employs. Our main objective is to obtain conditions that guarantee that all the trajectories of a given linear control system on time scales, that start outside the prescribed avoidance set, will never enter the set.

The paper is organized as follows. In Section 2 we briefly review the necessary calculus on time scales. In Section 3 we prove some results necessary in order to deal with delta-differential inclusions, which are considered in Section 4. Our main result is proved in Section 5. The obtained result is then applied in Section 6 to the case of linear control systems on time scales. We finish with Section 7, discussing a concrete example of application of our results. Necessary elements of delta-measurability and nonlinear theory on time scales are presented in Appendix.

2 Preliminaries

A *time scale* \mathbb{T} is an arbitrary nonempty closed subset of the set \mathbb{R} of real numbers. The standard cases comprise $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, and $\mathbb{T} = h\mathbb{Z}$ for $h > 0$. We assume that \mathbb{T} is a topological space with the topology induced from \mathbb{R} . For $t \in \mathbb{T}$ we define the *forward jump operator* $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, the *backward jump operator* $\rho : \mathbb{T} \rightarrow \mathbb{T}$ by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$, the *graininess function* $\mu : \mathbb{T} \rightarrow [0, \infty)$ by $\mu(t) := \sigma(t) - t$. Using these operators we can classify the points of the time scale: if $\sigma(t) > t$, then t is called *right-scattered*; if $\rho(t) < t$, then t is called *left-scattered*; if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense*; if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is *left-dense*. Function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in \mathbb{T} and its left-sided limits exist (finite) at left-dense points in \mathbb{T} . We write $f \in C_{rd}$. Function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called *regulated* provided its right-sided limits exist (finite) at all right-dense points of \mathbb{T} and

its left-sided limits exist (finite) at all left-dense points in \mathbb{T} . Function f is *piecewise rd-continuous*, denoted by $f \in C_{\text{prd}}$, if it is regulated and if it is rd-continuous at all, except possibly at finitely many, right-dense points $t \in \mathbb{T}$. Note that composition of a continuous function g with $f \in C_{\text{rd}}$ or $f \in C_{\text{prd}}$ is respectively rd-continuous or piecewise rd-continuous. Let \mathbb{T}^κ denote the set

$$\mathbb{T}^\kappa := \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty. \end{cases}$$

Definition 2.1 ([10]). Let $f : \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}^\kappa$. The *delta derivative* of f at t , denoted by $f^\Delta(t)$, is the real number (provided it exists) with the property that given any ε there is a neighborhood $U = (t - \delta, t + \delta) \cap \mathbb{T}$ (for some $\delta > 0$) such that $|(f(\sigma(t)) - f(s)) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|$ for all $s \in U$. We say that f is *delta differentiable* on \mathbb{T}^κ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.

A function f is *rd-continuously delta differentiable* (we write $f \in C_{\text{rd}}^1$) if f^Δ exists for all $t \in \mathbb{T}^\kappa$ and $f^\Delta \in C_{\text{rd}}$. A continuous function f is *piecewise rd-continuously delta differentiable* (we write $f \in C_{\text{prd}}^1$) if f is continuous and f^Δ exists for all, except possibly at finitely many $t \in \mathbb{T}^\kappa$, and $f^\Delta \in C_{\text{rd}}$.

Remark 2.1. If $\mathbb{T} = \mathbb{R}$, then for any $t \in \mathbb{R}$ we have $\sigma(t) = t = \rho(t)$ and the graininess function $\mu(t) \equiv 0$. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is delta differentiable at $t \in \mathbb{R}$ if and only if $f^\Delta(t) = \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s} = f'(t)$, i.e., if and only if f is differentiable in the ordinary sense at t . If $\mathbb{T} = \mathbb{Z}$, then for every $t \in \mathbb{Z}$ we have $\sigma(t) = t + 1$, $\rho(t) = t - 1$, and the graininess function $\mu(t) \equiv 1$. A function $f : \mathbb{Z} \rightarrow \mathbb{R}$ is always delta differentiable at every $t \in \mathbb{Z}$ with $f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = f(t + 1) - f(t) =: \Delta f(t)$.

Theorem 2.1 (Chain Rule [10]). *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \rightarrow \mathbb{R}$ is delta differentiable. Then $f \circ g : \mathbb{T} \rightarrow \mathbb{R}$ is delta differentiable and*

$$(f \circ g)^\Delta(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t)g^\Delta(t))dh \right\} g^\Delta(t).$$

A continuous function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called *pre-differentiable* with (the region of differentiation) D , provided $D \subset \mathbb{T}^\kappa$, and $\mathbb{T}^\kappa \setminus D$ is countable and contains no right-scattered elements of \mathbb{T} . For any regulated function f there

exists a function F that is pre-differentiable with the region of differentiation D such that $F^\Delta(t) = f(t)$ for all $t \in D$. Such F is called a pre-antiderivative of f . Then the *indefinite integral* of f is defined by $\int f(t)\Delta t := F(t) + C$, where C is an arbitrary constant. The *Cauchy integral* is defined by

$$\int_r^s f(t)\Delta t = F(s) - F(r)$$

for all $r, s \in \mathbb{T}^\kappa$. A function $F : \mathbb{T} \rightarrow \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \rightarrow \mathbb{R}$ provided $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^\kappa$. It can be shown that every rd-continuous function has an antiderivative [10].

Remark 2.2. If $\mathbb{T} = \mathbb{R}$, then $\int_a^b f(\tau)\Delta\tau = \int_a^b f(\tau)d\tau$ where the integral on the right hand-side is the usual Riemann integral. If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, and $a < b$, then $\int_a^b f(\tau)\Delta\tau = \sum_{t=\frac{a}{h}}^{\frac{b}{h}-1} f(th)h$.

We say that a function $v : \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is *right-increasing* (respectively *right-nondecreasing*) at a point $(t_0, x_0) \in \mathbb{T} \setminus \{\max \mathbb{T}\} \times \mathbb{R}^n$ provided:

- (i) if t_0 is right-scattered, then there is a neighborhood U_{x_0} of point x_0 such that $v(\sigma(t_0), x) > v(t_0, x_0)$ (respectively $v(\sigma(t_0), x) \geq v(t_0, x_0)$) for any $x \in U_{x_0}$;
- (ii) if t_0 is right-dense, then there is a neighborhood $U_{(t_0, x_0)}$ of point (t_0, x_0) such that $v(t, x) > v(t_0, x_0)$ (respectively $v(t, x) \geq v(t_0, x_0)$) for all $(t, x) \in U_{(t_0, x_0)}$ with $t > t_0$.

Similarly, we say that function $v : \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is *right-decreasing* (respectively *right-nonincreasing*) if above, in condition (i), $v(\sigma(t_0), x) < v(t_0, x_0)$ (respectively $v(\sigma(t_0), x) \leq v(t_0, x_0)$), and in condition (ii) $v(t, x) < v(t_0, x_0)$ (respectively $v(t, x) \leq v(t_0, x_0)$).

For each function $t \mapsto v(t, x(t)) \in C_{\text{rd}}^1$, let us define the operator ζ by

$$\zeta(v)(t, x(t)) := v(t + \mu(t), x(t) + \mu(t)x^\Delta(t)),$$

and operator \mathcal{D} by

$$\mathcal{D}(v)(t, x(t)) := \begin{cases} \frac{\zeta(v)(t, x(t)) - v(t, x(t))}{\mu(t)} & \text{for } \mu(t) \neq 0; \\ \frac{\partial v}{\partial t}(t, x(t)) + \frac{\partial v}{\partial x}(t, x(t))x^\Delta(t) & \text{for } \mu(t) = 0. \end{cases} \quad (1)$$

Note that if $\mu(t) \equiv 0$, then $\mathcal{D}(v)(t, x(t)) = \frac{d}{dt}v(t, x(t))$ and $\zeta(v) = v$ for any function v .

Proposition 2.1. *Let $t \in \mathbb{T}$.*

(i) *If $\mathcal{D}(v)(t, x(t)) \leq 0$, then v is a right-nonincreasing function of t .*

(ii) *If $\mathcal{D}(v)(t, x(t)) \geq 0$, then v is a right-nondecreasing function of t .*

Proof. We prove (i). Proof of (ii) is similar. Let us fix $t_0 \in \mathbb{T}$. If t_0 is right-scattered and $x \in U_{x_0}$, then $\mathcal{D}(v)(t, x(t)) \leq 0$ implies that

$$\frac{v(t_0 + \mu(t_0), x_0 + \mu(t_0)x_{|t=t_0}^\Delta(t)) - v(t, x_0)}{\mu(t_0)} \leq 0.$$

Since $\mu(t_0)$ is positive, then $\sigma(v)(t, x) \leq v(t_0, x_0)$.

Let t_0 be a right-dense point and $(t, x) \in U_{(t_0, x_0)}$. Let $\tau = t - t_0$ for every $t \in U_{(t_0, x_0)}$. Define $x_1(\tau) := t = \tau + t_0$ and $x_2(\tau) := v(t) = v(\tau + t_0)$, and note that the solution $v(t)$ for $t \in [t_0, +\infty)_{\mathbb{T}}$ of the Δ -differential equation $v^\Delta(t) = v(t, x(t))$, $v(t_0) = x_0$, can be obtain as solution $x_2(\tau)$, $\tau \geq 0$, of the following system of equations:

$$\begin{aligned} x_1^\Delta(\tau) &= 1, & x_1(0) &= t_0 \\ x_2^\Delta(\tau) &= v(x_1(\tau), x_2(\tau)), & x_2(0) &= x_0. \end{aligned}$$

Note that in this case $\mathcal{D}(v)(t, x(t))$ is nothing else as $\frac{d}{dt}(v(x_1(t), x_2(t)))$. Since $\mathcal{D}(v)(t, x(t)) \leq 0$, then $\frac{d}{dt}(v(x_1(t), x_2(t))) \leq 0$ and it follows that function $t \mapsto v(x_1(t), x_2(t))$ is a nonincreasing function of t . \square

3 Multifunctions on Time Scales

Let \mathbb{T} be an arbitrary time scale. If $t_0, t_1 \in \mathbb{T}$ and $t_0 \leq t_1$, then $[t_0, t_1]_{\mathbb{T}}$ denotes the intersection of the real closed interval $[t_0, t_1]$ with \mathbb{T} . Similar notation is used for open, half-open, or infinite intervals. Additionally, we will assume that interval $[t_0, t_1]_{\mathbb{T}}$ is nonempty.

Function f is said to be *lower rd-semicontinuous at point $t_0 \in \mathbb{T}$* if it is rd-continuous and for any $\varepsilon > 0$ there is a neighborhood U_{t_0} of t_0 in which $f(t) > f(t_0) - \varepsilon$. If $f(t_0) = -\infty$, then we regard f as lower semicontinuous at t_0 .

Lemma 3.1. *A finite lower rd-semicontinuous function f defined on a compact time scale interval is bounded from below.*

Proof. A lower rd-semicontinuous function is a regulated function. The result follows because every regulated function defined on a compact interval is bounded [10]. \square

Let us recall the definition of lexicographical ordering on \mathbb{R}^n . For any two distinct points $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ we write $x \prec y$ if $x_1 < y_1$ or there exists $i \in \{2, \dots, n\}$ such that $x_1 = y_1, \dots, x_{i-1} = y_{i-1}$ and $x_i < y_i$. Every compact set $K \subseteq \mathbb{R}^n$ has one first point $\xi = (\xi_1, \dots, \xi_n)$ with respect to the lexicographical order.

Let $t \in \mathbb{T}$ and let $t \rightarrow F(t) \subset \mathbb{R}^n$ be a multifunction with compact values. A *lexicographical selection* $t \rightarrow \xi(t) \in F(t)$, where $\xi(t)$ is the first point of the compact set $F(t)$ with respect to the lexicographical order, can be determined inductively as follows [12]. Let $\{e_1, \dots, e_n\}$ be the standard basis in \mathbb{R}^n . Define

$$\begin{aligned} \xi_1 &= \min_{x \in K} \langle e_1, x \rangle, & K_1 &= \{x \in K, \langle e_1, x \rangle = \xi_1\}, \\ \xi_2 &= \min_{x \in K_1} \langle e_2, x \rangle, & K_2 &= \{x \in K_1, \langle e_2, x \rangle = \xi_2\}, \\ & & & \vdots \\ \xi_n &= \min_{x \in K_{n-1}} \langle e_n, x \rangle, & K_n &= \{x \in K_{n-1}, \langle e_n, x \rangle = \xi_n\}. \end{aligned}$$

By induction, all sets $K_n \subseteq \dots \subseteq K_1 \subseteq K$ are compact, hence the components ξ_1, \dots, ξ_n are well defined. The set K_n contains the single point ξ , which precedes all other points of K in the lexicographical order.

Lemma 3.2. *Let \mathbb{T} be a time scale and $t \in \mathbb{T}$. Suppose that $t \rightarrow F(t) \subset \mathbb{R}^n$ is a bounded multifunction with closed graph, defined for t in a closed set $\mathcal{J}_{\mathbb{T}} \subseteq \mathbb{T}$ ($\mathcal{J}_{\mathbb{T}}$ denotes a time scale interval). Then for each vector $v \in \mathbb{R}^n$ the function $\varphi(t) := \min_{t \in F(t)} \langle v, t \rangle$ is lower rd-semicontinuous.*

Proof. Proof is similar to the classical one [12]. For fixed $\tau \in \mathcal{J}_{\mathbb{T}}$ consider a sequence $\{\tau_k\}$ of points $\tau_k \in \mathcal{J}_{\mathbb{T}}$ with $\tau_k \rightarrow \tau$, such that

$$\lim_{t \rightarrow \tau} \inf_{t \in \mathcal{J}_{\mathbb{T}}} \varphi(t) = \lim_{k \rightarrow \infty} \varphi(\tau_k).$$

Choose $t_k \in F(\tau_k) \subset \mathbb{R}^n$ such that $\varphi(\tau_k) = p \cdot t_k$, where p is any nonzero real number. Because $F(t)$ is a bounded multifunction with closed graph, then the obtained sequence of vectors, call them v , is uniformly bounded.

So, there exists a convergent subsequence $\{t_{k'}\}$ such that $t_{k'} \rightarrow \bar{t}$ and, because $F(t)$ has a closed graph, $\bar{t} \in F(t)$. Hence,

$$\varphi(\tau) = \min_{t \in F(t)} \langle v, t \rangle \leq \langle v, \bar{t} \rangle = \lim_{k \rightarrow \infty} \varphi(\tau_k) = \liminf_{t \rightarrow \tau} \mathcal{J}_{\mathbb{T}}.$$

□

Theorem 3.1. *Let \mathbb{T} be a time scale with $a, b \in \mathbb{T}$, $a < b$. Let $t \in \mathbb{T}$. Suppose that $t \rightarrow F(t) \subset \mathbb{R}^n$ is a bounded multifunction with closed graph, defined for $t \in [a, b]_{\mathbb{T}}$. Let $\xi(t) \in F(t)$ be the lexicographical selection for any $t \in \mathbb{T}$. Then the map $t \rightarrow \xi(t)$ is Δ -measurable.*

Proof. Let $\xi = (\xi_1, \dots, \xi_n)$. Because $\xi_1 = \min_{t \in F(t)} \langle e_1, t \rangle$, then Lemma 3.2 implies that the first component $t \rightarrow \xi_1(t)$ of the selection ξ is measurable. Let us assume that the first k components $\xi_1(\cdot), \dots, \xi_k(\cdot)$ are all measurable. Using Proposition A.1, Proposition A.4, and Remark A.1 in Appendix, we conclude that there exist countably many disjoint compact sets $\mathcal{J}_{k_{\mathbb{T}}}$ such that the maps ξ_1, \dots, ξ_k are all rd-continuous when restricted to each $\mathcal{J}_{k_{\mathbb{T}}}$. Let

$$t \rightarrow F_k(t) = \{\tau \in F(t) : \langle e_l, \tau \rangle = \xi_l(t), l = 1, \dots, k\}.$$

By construction, sets $F_k(t)$ are all nonempty and compact. Because multifunction $t \rightarrow F_k(t)$, after restriction to $\mathcal{J}_{k_{\mathbb{T}}}$, has a closed graph, it follows from Lemma 3.2 that function $\xi_{k+1}(t) = \min_{\tau \in F_k(t)} \langle e_l, \tau \rangle$ is lower rd-semicontinuous. Sets $\mathcal{J}_{k_{\mathbb{T}}}$ cover $[a, b]_{\mathbb{T}}$ almost everywhere, so $\xi_{k+1}(t)$ is Δ -measurable.

By the induction principle with respect to k , all components of $\xi(\cdot)$ are Δ -measurable. □

4 Delta-Differential Inclusions

Let us consider a control system

$$x^\Delta(t) = f(t, x(t), u(t)), \quad u \in \mathcal{U}, \quad (2)$$

where $\mathcal{U} = \{u(\cdot) \mid u(\cdot) \text{ is } \Delta\text{-measurable piecewise rd-continuous with } u(t) \in U \subset \mathbb{R}^m \text{ for all } t \in \mathbb{T}\}$ is the set of admissible controls. Suppose that the set $U \subset \mathbb{R}^m$ of control values is compact and that function $f : \mathbb{T} \times \Omega \times U \rightarrow \mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^n$, is rd-continuous with respect to the first variable and continuously differentiable with respect to the second variable $x \in \mathbb{R}^n$.

We say that a rd-continuous function $f : [t_0, t_1]_{\mathbb{T}} \rightarrow \mathbb{R}^n$ is *absolutely rd-continuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that the following holds (cf. [13]): if $[s_i, t_i]_{\mathbb{T}}$ is any finite collection of disjoint time scale intervals such that $\sum_{i=1}^n |t_i - s_i| \leq \delta$, then $\sum_{i=1}^n |f(t_i) - f(s_i)| \leq \varepsilon$. An absolutely rd-continuous function $x(\cdot)$ defined on some time scale interval $[t_0, t_1]_{\mathbb{T}}$ is a solution of (2) if its graph $\{(t, x(t)) : t \in [t_0, t_1]_{\mathbb{T}}\}$ is entirely contained in Ω , and if there exists a measurable control u , taking values inside U , such that $x^\Delta(t) = f(t, x(t), u(t))$ for almost every $t \in [t_0, t_1]_{\mathbb{T}}$. Let us recall that by *trajectory* of the system (2) from x_0 corresponding to the control $u \in \mathcal{U}$ we mean the function

$$x = \psi(t_0, \cdot, x_0, u) : [t_0^u, t_1^u]_{\mathbb{T}} \rightarrow \mathbb{R}^n \quad (3)$$

such that x is the unique solution of the initial value problem $x^\Delta(t) = f_u(t, x(t))$, $x(t_0) = x_0$, provided it is defined on the interval $[t_0, t_1]_{\mathbb{T}} \subseteq \mathcal{I}_{max}$ (\mathcal{I}_{max} described in Theorem A.2) for all $t \in [t_0, t_1]_{\mathbb{T}}$ and $x(t) \in \Omega$.

In connection with (2) let us take the multifunction

$$F(t, x) = \{f(t, x, \omega) : \omega \in U\},$$

where $U \subset \mathbb{R}^m$ is the set of control values, and consider the Δ -differential inclusion

$$x^\Delta(t) \in F(t, x(t)). \quad (4)$$

We extend the classical Filippov's theorem [14] to time scales as follows.

Theorem 4.1. *An absolutely rd-continuous function $x : [t_0, t_1]_{\mathbb{T}} \rightarrow \mathbb{R}^n$ is a trajectory of (2) if and only if it satisfies (4) almost everywhere.*

Proof. The proof mimics the one given in [12] for the continuous-time case. It is obvious that every solution of (2) is a solution of (4). We prove the reverse implication. Let $x(\cdot)$ be a solution of (4). For a fixed element $\bar{\omega} \in U$, let

$$W(t) = \begin{cases} \{\omega \in U : f(t, x(t), \omega) = x^\Delta(t)\} & \text{if } x^\Delta(t) \in F(t, x(t)) \\ \{\bar{\omega}\} & \text{otherwise.} \end{cases}$$

Note that multifunction $W(t)$ is compact. Let us define a control u in such a way that $u(t)$ is the first element of the set $W(t)$ with respect to the lexicographical order. Note that such control exists since the multifunction $W(t)$ is compact. Because $W(t) = \bar{\omega}$ holds only on a set of Δ -measure zero,

i.e., when $x(\cdot)$ doesn't exist or $x^\Delta(t) \neq F(t, x(t))$, then $x^\Delta(t) = f(t, x(t), u(t))$ for almost all $t \in [t_0, t_1]_{\mathbb{T}}$. Let us choose a sequence of disjoint compact intervals $\mathcal{J}_{k_{\mathbb{T}}} \subseteq [a, b]_{\mathbb{T}}$, $k \in \mathbb{N}$, such that $\mu_{\Delta} \{[a, b] \setminus \bigcup_{k=1}^{\infty} \mathcal{J}_k\} = 0$ and the Δ -derivative x^Δ is well defined and rd-continuous when restricted to $\mathcal{J}_{k_{\mathbb{T}}}$. Hence, the multifunction $W|_{\mathcal{J}_{k_{\mathbb{T}}}}$ has closed graph. Theorem 3.1 implies that for any $t \in \mathcal{J}_{k_{\mathbb{T}}}$ the lexicographical selection $t \rightarrow u(t) \in W(t)$ is Δ -measurable. Since the sets $\mathcal{J}_{k_{\mathbb{T}}}$ cover $[a, b]_{\mathbb{T}}$ almost everywhere, the selection $u : [a, b]_{\mathbb{T}} \rightarrow U$ is Δ -measurable. \square

5 Leitmann's Avoidance Strategies on Time Scales

We are now in conditions to extend Leitmann's avoidance strategies to any time t from an arbitrary time scale \mathbb{T} .

Let $p^i : \mathbb{T} \times \mathbb{R}^n \rightarrow U_i$, $i = 1, 2$, where U_i is a nonempty subset of the space \mathbb{R}^{d_i} , be strategies belonging to given classes of possibility set valued functions \mathcal{U}_i with control values u^i ranging in given sets U_i . We allow state- and time-dependent constraints, i.e., $U_i = U_i(t, x)$. Then $u^i \in p^i(t, x) \subseteq U_i \subseteq \mathbb{R}^{d_i}$.

Let us consider a function $f : \mathbb{T} \times \mathbb{R}^n \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \rightarrow \mathbb{R}^n$ and a set-valued function

$$F(t, x) := \{z : z = f(t, x, u^1, u^2) \text{ where } u^i \in p^i(t, x)\}.$$

By a *dynamical system* on the time scale \mathbb{T} we will mean the Δ -differential relation

$$x^\Delta(t) \in F(t, x(t)), \quad x(t_0) = x_0, \quad (5)$$

$t_0 \in \mathbb{T}$. From Filippov's theorem on time scales (Theorem 4.1) it follows that, for the given initial condition $x(t_0) = x_0$, the solution of the Δ -differential inclusion (5) is an absolute rd-continuous function

$$x : [t_0, t_1]_{\mathbb{T}} \rightarrow \mathbb{R}^n, \quad (6)$$

satisfying (5) almost everywhere for $t \in [t_0, t_1]_{\mathbb{T}}$. For x given by (6) one has

$$\begin{aligned} \zeta(v)(t, x(t)) &= v(t + \mu(t), x(t) + \mu(t)x^\Delta(t)) \\ &= v(t + \mu(t), x(t) + \mu(t)f(t, x(t), u^1(t), u^2(t))) \end{aligned} \quad (7)$$

for some u^1 and u^2 . We will write (7) as $\zeta(v)(t, x(t), u^1(t), u^2(t))$. Similarly, (1) will be written as

$$\begin{aligned} & \mathcal{D}(v)(t, x(t), u^1(t), u^2(t)) \\ &= \begin{cases} \frac{\zeta(v)(t, x(t), u^1(t), u^2(t)) - v(t, x(t))}{\mu(t)} & \text{for } \mu(t) \neq 0; \\ \frac{\partial v}{\partial t}(t, x(t)) + \frac{\partial v}{\partial x}(t, x(t))f(t, x(t), u^1(t), u^2(t)) & \text{for } \mu(t) = 0. \end{cases} \quad (8) \end{aligned}$$

Sometimes it may be convenient to restrict x by $x|_\Lambda$ where Λ is an open set or the closure of an open set in \mathbb{R}^n . Then $(t_0, x_0) \in \mathbb{T} \times \Lambda$.

Let $\mathcal{T} \subseteq \Lambda$ denote a set into which no solution of (5) must enter for some $p^1(\cdot) \in \mathcal{U}_1$ no matter what $p^2(\cdot) \in \mathcal{U}_2$. The set \mathcal{T} is called an *antitarget set*. Let us consider two sets:

1. a closed subset $\mathcal{A} \subseteq \Lambda$ such that $\mathcal{T} \subset \mathcal{A}$;
2. the closure Λ_ε of an open subset of Λ such that $\mathcal{A} \subset \Lambda_\varepsilon$.

The set \mathcal{A} will be called the *avoidance set* while the set $\Lambda_\varepsilon \setminus \mathcal{A}$ will be called the *safety zone*. Note that the avoidance set can be any set containing the antitarget set \mathcal{T} .

PROBLEM: Determine an *avoidance strategy* $p^1(\cdot) \in \mathcal{U}_1$ such that, given $(t_0, x_0) \in \mathbb{T} \times \Lambda_\varepsilon$, no solution of (5) intersects \mathcal{A} no matter what $p^2(\cdot) \in \mathcal{U}_2$.

Denote $\Omega_\mathcal{A} = \mathbb{T}^\kappa \times \Lambda_\mathcal{A}$. Then an *attainable set of motions* $K(t, t_0, x_0)$ from (t_0, x_0) at time $t \in [t_0; \infty)_\mathbb{T}$ for the given $p^1(\cdot) \in \mathcal{U}_1$ is defined in the following way:

$$K(t, t_0, x_0) := \{x(t) : \text{given } (t_0, x_0) \in \Omega_\mathcal{A}, \text{ given } p^1(\cdot) \in \mathcal{U}_1 \text{ for all } p^2(\cdot) \in \mathcal{U}_2\}.$$

The *funnel of motions* \mathcal{U}_2 from (t_0, x_0) is defined by

$$K([t_0; \infty)_\mathbb{T}, t_0, x_0) := \bigcup_{t \in [t_0; \infty)_\mathbb{T}} K(t, t_0, x_0),$$

and

$$K(\Omega_\mathcal{A}, \mathbb{T}) := \bigcup_{(t_0, x_0) \in \Omega_\mathcal{A}} K([t_0; \infty)_\mathbb{T}, t_0, x_0).$$

For the given dynamical system (5) and \mathcal{U}_i , $i = 1, 2$, a set \mathcal{A} is called *avoidable* if and only if there are $p^1(\cdot) \in \mathcal{U}_1$ and $\Omega_{\mathcal{A}} \neq \partial\Lambda_\varepsilon \cap \partial\mathcal{A} \cap \text{int}\Lambda$ such that

$$K(\Omega_{\mathcal{A}}, \mathbb{T}) \cap \mathcal{A} = \partial\Lambda_\varepsilon \cap \partial\mathcal{A} \cap \text{int}\Lambda. \quad (9)$$

The condition (9) implies that

$$K[\mathbb{T}, \mathbb{T} \times (\Lambda \setminus \mathcal{A})] \cap \mathcal{A} = \partial\Lambda_\varepsilon \cap \partial\mathcal{A} \cap \text{int}\Lambda$$

and this means global avoidance.

Theorem 5.1. *A given set \mathcal{A} is avoidable if there exist a set $\Omega_{\mathcal{A}} \neq \emptyset$, a strategy $p^1(\cdot) \in \mathcal{U}_1$, and a rd-continuous function $V : S \rightarrow \mathbb{R}$, where S is an open subset of $\bar{\Omega}_{\mathcal{A}}$, such that for all $(t, x(t)) \in \Omega_{\mathcal{A}}$:*

- (i) $V(t, x(t)) > V(t_1, x^1)$ for all $x^1 \in \partial\mathcal{A}$ and $t_1 \geq t$ with $t_1, t \in \mathbb{T}^\kappa$;
- (ii) for all $u^1(t) \in p^1(t, x(t))$, where $\mathcal{D}(V)(t, x(t), u^1(t), u^2(t)) \geq 0$ for all $u^2(t) \in U_2$, $\tilde{p}^1(\cdot) = p^1(\cdot)|_{\Omega_{\mathcal{A}}}$.

Proof. Let $t_0 \in \mathbb{T} \setminus \{\max \mathbb{T}\}$. Assume that for some $(t_0, x_0) \in \Omega_{\mathcal{A}}$ there is $t_2 \in \mathbb{T}$, $t_2 > t_0$, such that $K(t_0, t_2, x_0) \cap \mathcal{A} \neq \emptyset$. Then, by (i), there is $t_1 \in (t_0, t_2]_{\mathbb{T}}$ and $x^1 \in K(t_0, t_2, x_0) \cap \partial\mathcal{A}$ such that $V(t_0, x_0) > V(t_1, x^1)$. Note that along the trajectories of the system (5) we have $\mathcal{D}(V)(t, x(t), u(t))$ given in accordance with (8), with $u(t) = (u^1(t), u^2(t))$. Thus, it is enough to show that for $(t, x) \in \Omega_{\mathcal{A}}$ such that $x \neq x_0$ and $t \neq t_0$, $t \in \mathbb{T}^\kappa$, there is a $\tau \in \mathbb{T}$ and an admissible control u such that for the trajectory $\psi(t_0, t, x_0, u)$ of the system Λ we have $V(t, \psi(t_0, t, x_0, u)) \geq V(t, x)$ for all $t \in (t_0, \tau]_{\mathbb{T}}$ and $V(\tau, \psi(t_0, \tau, x_0, u)) > V(\tau, x)$.

Let $(t, x) \in \Omega_{\mathcal{A}}$ be such that $x \neq x_0$ and $t \neq t_0$ with $t \in \mathbb{T}^\kappa$. Let us take two admissible controls u^1 and u^2 for the system (5). Then $\psi(t_0, \cdot, x_0, u)$ given by (3) is a trajectory corresponding to the control $u = (u^1, u^2)$. Let us consider the maximal solution of the initial value problem $x^\Delta(t) = f_u(t, x(t))$, $x(t_0) = x_0$. This is defined on some interval $[t_0, \tau)_{\mathbb{T}}$ with $\tau \leq +\infty$. If $\tau = +\infty$, then condition (i) implies that $V(t, \psi(t_0, t, x_0, u))$ is right-nondecreasing as long as $(t, \psi(t_0, t, x_0, u))$ remains in the set $\Omega_{\mathcal{A}}$. For this it is sufficient that it remains in the set $Z := \{(t, x) : V(t, x) \geq \varepsilon\}$ for some choice of ε . Assume that there is some $\tilde{t} > 0$ in the interval $\mathcal{I}_{\mathbb{T}}$ such that $V(\tilde{t}, \psi(t_0, \tilde{t}, x_0, u)) \leq \varepsilon$. Because of continuity of V , there is such a first \tilde{t} . Thus, we may assume that $V(t, x(t)) > \varepsilon$ for all $t \in [t_0, \tilde{t})_{\mathbb{T}}$. So $t \mapsto V(t, x(t))$ is right-nondecreasing on

$[t_0, \tilde{t}]_{\mathbb{T}}$, which implies that $V(\tilde{t}, \psi(t_0, \tilde{t}, x_0, u)) \geq V(t, x) > \varepsilon$. We arrived to a contradiction, and we conclude that ψ remains in the compact set Z for all $t \in \mathcal{I}_{\mathbb{T}}$. By Proposition A.6 in Appendix, $\mathcal{I}_{\mathbb{T}} = [t_0, +\infty)_{\mathbb{T}}$ (as desired). Thus, the trajectory is defined for all $t \in \mathbb{T}$ and V is right-nondecreasing. \square

6 Linear Control Systems on Time Scales

Let \mathbb{T} be an unbounded time scale. Let us consider the case of a dynamical system (5) in which its solution $x^\Delta(t) \in f(t, x(t), p^1(t, x(t)), p^2(t, x(t)))$ is linear, i.e., described by a linear equation of the form

$$x^\Delta(t) = Ax(t) + Bu^1(t) + Cu^2(t) \quad (10)$$

where $x \in \Lambda \subseteq \mathbb{R}^n$, $u^i(t) \in U_i \subseteq \mathbb{R}^{d_i}$ for $i = 1, 2$, and A, B, C are constant matrices. Equation (10) has a unique forward solution [15].

Suppose that matrix A is stable, i.e., satisfies a Lyapunov equation. The theory of Lyapunov stability on time scales for linear control systems is studied in detail in [16, 17]. Let Q be a symmetric constant matrix such that $Q \in C_{\text{rd}}(\mathbb{T}; \mathbb{R}^{n \times n})$. A quadratic Lyapunov function is given by

$$x^T(t)Qx(t), \quad t \in [t_0; \infty)_{\mathbb{T}}, \quad (11)$$

with delta derivative

$$\begin{aligned} [x^T(t)Qx(t)]^\Delta &= [Ax(t)]^T Qx(t) + x(\sigma(t))QAx(t) \\ &= x^T(t)[A^T Q + QA + \mu(t)A^T QA]x(t). \end{aligned}$$

The matrix dynamic equation that is obtained by delta differentiating (11) with respect to t is given by

$$A^T Q + QA + \mu(t)A^T QA = M, \quad M = M^T. \quad (12)$$

Theorem 6.1 ([17]). *If the $n \times n$ matrix A has all eigenvalues in the corresponding Hilger circle for every $t \in [t_0, \infty)_{\mathbb{T}}$, then for each $t \in \mathbb{T}$ there exists some time scale \mathbb{S} such that integration over $I := [0; \infty)_{\mathbb{S}}$ yields a unique solution to the Lyapunov matrix equation (12):*

$$Q(t) = \int_I e_{A^T}^T(s, 0) M e_A(s, 0) \Delta s. \quad (13)$$

Moreover, if M is positive definite, then $Q(t)$ is positive definite for all $t \in [t_0, \infty)_{\mathbb{T}}$.

Let us choose a matrix V in such a way that $V(t, x) = x^T(t)Qx(t)$. Then, if the avoidance set $\mathcal{A} = \{x : x^T Px \leq \text{const}\}$, i.e., if it is a ball in \mathbb{R}^n , then the condition (i) of Theorem 5.1 is satisfied. Moreover, let us take a constant matrix D in such a way that $C = BD$ and define

$$U_i := \{u_i : \|u_i\| \leq \alpha_i, i = 1, 2\}$$

with α_i positive constants chosen in such a way that $\alpha_1 \geq \|D\|\alpha_2$, where $\|\cdot\|$ denotes the Euclidian norm. Then the condition (ii) of Theorem 5.1 is met. In this case the avoidance strategy is given by

$$\tilde{p}^1(t, x) = \frac{B^T Q x}{\|B^T Q x\|} \alpha_1$$

for any (t, x) that does not belong to the set $\mathbb{T} \times \{x : B^T Q x = 0\}$. If $(t, x) \in \mathbb{T} \times \{x : B^T Q x = 0\}$, then \tilde{u}^1 may take any admissible value, i.e., $\tilde{p}^1(t, x) = U_1$.

Remark 6.1. Let us note that, because of (13), the avoidance strategy depends on the time scale \mathbb{T} . For $\mathbb{T} = \mathbb{R}$ the equation (12) is nothing else as the classical Lyapunov equation for the time-invariant system. Similarly for $\mathbb{T} = \mathbb{Z}$.

If the matrix $-A$ is not stable, but pair $(-A, -B)$ is stabilizable and graininess function is bounded, then there exist a constant matrix K such that $-A - KB$ is stable (see [18]) and avoidance strategy is of the form

$$\tilde{p}^1(t, x) = Kx + \frac{B^T Q x}{\|B^T Q x\|} \|D\| \alpha_2. \quad (14)$$

For the strategy (14) to be admissible, $U_1 = U_1(t, x)$ must be such that $\tilde{p}^1(t, x) \subseteq U_1$ for all $(t, x) \in \Omega_{\mathcal{A}}$.

7 Illustrative Example

Let us consider a linear control system

$$x^\Delta(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u^1(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u^2(t)$$

with

$$U_2 = \{u^2 : |u^2| \leq 1\}.$$

Note that matrix $-A$ is not stable, but pair $(-A, -B)$ is stabilizable. Then $K = \begin{bmatrix} -1 & 1 \end{bmatrix}$. Let us choose as matrix M the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then, matrix Q , that solves equation (12), is of form

$$Q = \begin{bmatrix} \frac{2(2\mu + \mu^2 + 3)}{6\mu + 4 + 3\mu^2 + \mu^3} & -\frac{\mu^2 + \mu + 2}{6\mu + 4 + 3\mu^2 + \mu^3} \\ -\frac{\mu^2 + \mu + 2}{6\mu + 4 + 3\mu^2 + \mu^3} & \frac{\mu^2 + 4 + \mu}{6\mu + 4 + 3\mu^2 + \mu^3} \end{bmatrix}. \quad (15)$$

The nonlinear part of $\tilde{p}^1(\cdot)$ is

$$\text{sign} \left(-\frac{\mu^2 + \mu + 2}{6\mu + 4 + 3\mu^2 + \mu^3} x_1 + \frac{\mu^2 + 4 + \mu}{6\mu + 4 + 3\mu^2 + \mu^3} x_2 \right).$$

Provided $\mathcal{A} = \{x : x^T Q \leq a, \text{ with } a \text{ any positive real constant}\}$, any avoidance control is given by

$$\tilde{p}^1(t, x) = -x_1 + x_2 + \text{sign} \left(-\frac{\mu^2 + \mu + 2}{6\mu + 4 + 3\mu^2 + \mu^3} x_1 + \frac{\mu^2 + 4 + \mu}{6\mu + 4 + 3\mu^2 + \mu^3} x_2 \right)$$

for all

$$(t, x) \notin \left\{ (t, x) : -\frac{\mu^2 + \mu + 2}{6\mu + 4 + 3\mu^2 + \mu^3} x_1 + \frac{\mu^2 + 4 + \mu}{6\mu + 4 + 3\mu^2 + \mu^3} x_2 = 0 \text{ and } t \in \mathbb{T} \right\},$$

i.e., for all

$$(t, x) \notin \{(t, x) : (\mu^2 + \mu + 2)x_1 = (\mu^2 + 4 + \mu)x_2 \text{ and } t \in \mathbb{T}\}.$$

Because the constrain set U_1 depends on the choice of the safety zone $\Delta_{\mathcal{A}}$, one may choose $\Delta_\varepsilon = \{x : x^T Q x \leq a + \varepsilon \text{ for } \varepsilon > 0\}$, so U_1 must be such that $\tilde{p}^1(t, x) \subseteq U_1$ for all $x \in \Delta_{\mathcal{A}} = \Delta_\varepsilon \setminus \mathcal{A}$ and $t \in \mathbb{T}$.

Note that if $\mathbb{T} = \mathbb{R}$ then matrix Q given by (15) is nothing else as

$$Q = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$$

and avoidance control is given by $\tilde{p}^1(t, x) = -x_1 + x_2 + \text{sign}(-\frac{1}{2}x_1 + x_2)$ for all $(t, x) \notin \{(t, x) : x_1 = 2x_2 \text{ and } t \in \mathbb{R}\}$. Hence, for $\mathbb{T} = \mathbb{R}$ we have nothing else as the result of the example given in [6].

If $\mathbb{T} = \mathbb{Z}$, then matrix

$$Q = \begin{bmatrix} \frac{6}{7} & -\frac{2}{7} \\ -\frac{2}{7} & \frac{3}{7} \end{bmatrix}.$$

Thus, avoidance control is $\tilde{p}^1(t, x) = -x_1 + x_2 + \text{sign}(-\frac{2}{7}x_1 + \frac{3}{7}x_2)$ for all $(t, x) \notin \{(t, x) : 2x_1 = 3x_2 \text{ and } t \in \mathbb{Z}\}$.

Let us consider now the "impulsive" time scale $\mathbb{T} = \mathbb{P}_{1,2} := \bigcup_{k=0}^{\infty} [3k, 3k+1]$. The graininess function takes the form

$$\mu(t) = \begin{cases} 0 & \text{if } t \in \bigcup_{k=0}^{\infty} [3k, 3k+1); \\ 2 & \text{if } t \in \bigcup_{k=0}^{\infty} \{3k+1\} \end{cases}$$

and

$$Q = \begin{cases} \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} & \text{if } t \in \bigcup_{k=0}^{\infty} [3k, 3k+1); \\ \begin{bmatrix} \frac{11}{18} & -\frac{2}{9} \\ -\frac{2}{9} & \frac{5}{18} \end{bmatrix} & \text{if } t \in \bigcup_{k=0}^{\infty} \{3k+1\}. \end{cases}$$

Then, avoidance control is given by

$$\tilde{p}^1(t, x) = \begin{cases} -x_1 + x_2 + \text{sign}(-\frac{1}{2}x_1 + x_2) & \text{if } t \in \bigcup_{k=0}^{\infty} [3k, 3k+1); \\ -x_1 + x_2 + \text{sign}(-\frac{2}{9}x_1 + \frac{5}{18}x_2) & \text{if } t \in \bigcup_{k=0}^{\infty} \{3k+1\} \end{cases}$$

for all

$$(t, x) \notin \begin{cases} \{(t, x) : x_1 = 2x_2 \text{ and } t \in \bigcup_{k=0}^{\infty} [3k, 3k+1)\}; \\ \{(t, x) : 4x_1 = 5x_2 \text{ and } t \in \bigcup_{k=0}^{\infty} \{3k+1\}\}. \end{cases}$$

A Appendix

Elements of Δ -Measures on Time Scales

The notions of Δ -measurable set and Δ -measurable function are studied in [19, 20]. Let us consider a set $\mathcal{F} = \{[a, b]_{\mathbb{T}} : a, b \in \mathbb{T}, a \leq b\}$. The interval $[a, a]_{\mathbb{T}}$ is understood as the empty set. Let $m_1 : \mathcal{F} \rightarrow [0, \infty)$ be a set of functions that assigns to each interval $[a, b]_{\mathbb{T}} \in \mathcal{F}$ its length: $m_1([a, b]_{\mathbb{T}}) = b - a$.

Using the pair $(\mathcal{F}; m_1)$, one can generate an outer measure m_1^* on the family of all subsets of \mathbb{T} as follows. Let E be any subset of \mathbb{T} . If there exists

at least one finite or countable system of intervals $V_j \in \mathcal{F}$, $j \in \mathbb{N}$, such that $E \subset \bigcup_j V_j$, then we put

$$m_1^*(E) = \inf \sum_j m_1(V_j)$$

where the infimum is taken over all coverings of E by a finite or countable system of intervals $V_j \subset \mathcal{F}$. If there is no such covering of E , then we put $m_1^*(E) = \infty$. A set $A \subset \mathbb{T}$ is said to be Δ -measurable if equality

$$m_1^*(E) = m_1^*(E \cap A) + m_1^*(E \cap (\mathbb{T} \setminus A))$$

holds true for any subset E of \mathbb{T} . Defining the family

$$\mathcal{M}(m_1^*) = \{A \subset \mathbb{T} : A \text{ is } \Delta\text{-measurable}\},$$

the *Lebesgue Δ -measure*, denoted by μ_Δ , is the restriction of m_1^* to $\mathcal{M}(m_1^*)$.

Proposition A.1 ([20]). *If set E is Lebesgue measurable, then set $E \cap \mathbb{T}$ is Δ -measurable.*

Let $\tilde{\mathbb{R}} := [-\infty, +\infty]$. We say that function $f : \mathbb{T} \rightarrow \tilde{\mathbb{R}}$ is Δ -measurable if for every $\alpha \in \mathbb{R}$ the set $f^{-1}([-\infty, \alpha)) = \{t \in \mathbb{T} : f(t) < \alpha\}$ is Δ -measurable.

Proposition A.2 ([20]). *If f is rd-continuous, then f is Δ -measurable.*

Proposition A.2 and properties of rd-continuous and continuous functions on time scales implies that if f is a continuous function defined on \mathbb{T} , then f is Δ -measurable. Moreover, if an rd-continuous function f is defined on a Δ -measurable set $E \subseteq \mathbb{T}$, then f is a Δ -measurable function.

Proposition A.3 ([20]). *Let f be defined on a Δ -measurable subset E of \mathbb{T} . Function f is Δ -measurable if the set of all right-dense points of E , where f is discontinuous, is a set of Δ -measure zero.*

Let us define the function

$$\tilde{f}(t) := \begin{cases} f(t) & \text{if } t \in \mathbb{T} \\ f(t_i) & \text{if } t \in (t_i, \sigma(t_i)) \end{cases} \quad (16)$$

for some $i \in I \subseteq \mathbb{N}$ and $\{t_i\}_{i \in I} = \{t \in \mathbb{T} : t < \sigma(t)\}$. Then, f is Δ -measurable if and only if \tilde{f} is Lebesgue measurable.

Proposition A.4 ([19]). Assume that $f : \mathbb{T} \rightarrow \tilde{\mathbb{R}}$ and $\tilde{f} : [a, b]_{\mathbb{R}} \rightarrow \tilde{\mathbb{R}}$ is the extension of f to $[a, b]_{\mathbb{R}}$, defined by (16). Then f is Δ -measurable if and only if \tilde{f} is Lebesgue measurable.

Let us recall the classical Luzin's theorem.

Theorem A.1 ([21]). Function $f : [a, b]_{\mathbb{R}} \rightarrow \mathbb{R}$ is Lebesgue measurable if and only if, given $\varepsilon > 0$, there is a continuous function $\varphi : [a, b]_{\mathbb{R}} \rightarrow \mathbb{R}$ such that the Lebesgue measure of the set $\{x : f(x) \neq \varphi(x)\}$ is strictly less than ε .

As an immediate consequence of Proposition A.4 and Luzin's Theorem A.1, we obtain an extension of Luzin's theorem to time scales.

Corollary A.1. Assume that $f : \mathbb{T} \rightarrow \tilde{\mathbb{R}}$. Then f is Δ -measurable if and only if, given $\varepsilon > 0$, there is a rd-continuous function $\varphi : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ such that the Δ -Lebesgue measure of the set $\{x : f(x) \neq \varphi(x)\}$ is strictly less than ε .

Remark A.1. Function $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}^n$ is Δ -measurable if and only if there exists a sequence of disjoint compact subsets $\mathcal{J}_k \subset [a, b]_{\mathbb{T}}$ with $\mu_{\Delta}\{[a, b] \setminus \bigcup_{k=1}^{\infty} \mathcal{J}_k\} = 0$ such that $f|_{\mathcal{J}_k}$ is rd-continuous.

Nonlinear Δ -Differential Equations on Time Scales

Let us recall [10] that function $f : \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called

1. *rd-continuous*, if g defined by $g(t) = f(t, x(t))$ is rd-continuous for any continuous function $x : \mathbb{T} \rightarrow \mathbb{R}^n$;
2. *bounded* on a set $S \subseteq \mathbb{T} \times \mathbb{R}^n$, if there exist constants m and M such that $m \leq f(t, x) \leq M$ for all $(t, x) \in S$.

Theorem A.2 ([10]). Let $t_0 \in \mathbb{T}$, $x_0 \in \mathbb{R}^n$, $a > 0$ with $\inf \mathbb{T} \leq t_0 - a$, and $\sup \mathbb{T} \geq t_0 + a$. Put $\mathcal{I}_a = (t_0 - a, t_0 + a)_{\mathbb{T}}$ and $\mathcal{V}_b = \{x \in \mathbb{R}^n : |x - x_0| < b\}$. Let $f : \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be rd-continuous and regressive. Suppose that for each $(t, x) \in \mathbb{T} \times \mathbb{R}^n$ there exists a neighborhood $\mathcal{I}_a \times \mathcal{V}_b$ such that f is bounded on $\mathcal{I}_a \times \mathcal{V}_b$ and such that the Lipschitz condition

$$|f(t, x_1) - f(t, x_2)| \leq L(t, x)|x_1 - x_2| \text{ for all } (t, x_1), (t, x_2) \in \mathcal{I}_a \times \mathcal{V}_b$$

holds, where $L(t, x) > 0$. Then, the initial value problem

$$x^{\Delta} = f(t, x), \quad x(t_0) = x_0 \tag{17}$$

has exactly one maximal solution $\tilde{\lambda} : \mathcal{I}_{max} \rightarrow \mathbb{R}^n$, and the maximal interval of existence $\mathcal{I}_{max} = \mathcal{I}_{max}(t_0, x_0)$ is open.

Theorem A.3 ([10]). *Suppose that assumptions of Theorem A.2 are satisfied, and assume that there exist positive and continuous functions p and q with*

$$|f(t, x)| \leq p(t)|x| + q(t) \text{ for all } (t, x) \in \mathbb{T} \times \mathbb{R}^n.$$

Then each solution of $x^\Delta = f(t, x)$ exists on all of \mathbb{T} .

Proposition A.5. *For any compact $\mathcal{K} \subseteq \mathbb{R}^n$ there locally exists a continuous function γ such that*

$$|f(t, x)| \leq \lambda(t) \text{ for all } (t, x) \in \mathcal{I}_a \times \mathcal{K}.$$

Proof. The proof is similar to the one for the real time case (see [22]). Let us note that for any $x \in \mathcal{V}_b$ and $t \in \mathcal{I}_a$, \mathcal{V}_b and \mathcal{I}_a such as in assumptions of Theorem A.2, we have

$$|f(t, x)| \leq |f(t, x_0)| + |f(t, x) - f(t, x_0)| \leq p(t)|x| + q(t).$$

Put $\lambda_{x_0} = p(t)|x| + q(t)$. By compactness of the set \mathcal{K} there exist a finite subcover, corresponding to sets $\mathcal{V}_{b_1}, \dots, \mathcal{V}_{b_l}$, centered at x_1, \dots, x_l . Taking $\lambda(t) := \max\{\lambda_{x_1}, \dots, \lambda_{x_l}\}$ we arrive to the intended conclusion. \square

Note that function λ defined above is integrable on time scales.

Proposition A.6. *Suppose that assumptions of Theorem A.2 are satisfied, and assume that it is known that there is a compact subset $\mathcal{K} \subseteq \mathbb{R}^n$ such that the maximal solution ψ of (17) satisfies $\psi(t) \in \mathcal{K}$ for all $t \in \mathcal{J}_{\mathbb{T}} \subseteq \mathcal{I}_a$ where $\mathcal{J}_{\mathbb{T}}$ is open relative to \mathcal{I}_a . Then,*

$$\mathcal{J}_{\mathbb{T}} = [t_0, +\infty)_{\mathbb{T}} \cap \mathcal{I}_a.$$

Proof. Proof is the same as the one given for the real time case (see [22]). \square

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